Can you add polynomials with different exponents

Learning Outcomes Add polynomials Use horizontal and vertical organization to add polynomials Find the opposite of a polynomial Subtract polynomials using both horizontal and vertical organization Multiply Polynomials Find the product of monomials Find the product of polynomials and monomials Find the product of two binomials Multiply Binomials Apply the FOIL method to multiply two binomials Use a table to multiply two binomials Simplify the product of two binomials given a wide variety of variables, constants, signs, and arrangement of terms in the binomial Divide Polynomials Divide a binomial by a monomial Divide a trinomial by a monomial Apply polynomial long division to divide by a binomial Adding and subtracting polynomials may sound complicated, but it's really not much different from the addition and subtraction that you do every day. The main thing to remember is to look for and combine like terms. You can add two (or more) polynomials as you have added algebraic expressions. You can remove the parentheses and combine like terms. When you are adding polynomials that have subtraction, it is important to remember to keep the sign on each term as you are collecting like terms.   The next example will show you how to regroup terms that are subtracted when you are collecting like terms. The above examples show addition of polynomials horizontally, by reading from left to right along the same line. Some people like to organize their work vertically instead, because they find it easier to be sure that they are combining like terms. The example below shows this "vertical" method of adding polynomials: Sometimes in a vertical arrangement, you can line up every term beneath a like term, as in the example above. But sometimes it isn't so tidy. When there isn't a matching like term for every term, there will be empty places in the vertical arrangement. You may be thinking, how is this different than combining like terms, which we did in the last section? The answer is, it's not really. We just added a layer to combining like terms by adding more terms to combine. :) Polynomials are a useful tool for describing the behavior of anything that isn't linear, and sometimes you may need to add them. In the following video, you will see more examples of combining like terms by adding polynomials. Find the opposite of a polynomial Opposites When you are solving equations, it may come up that you need to subtract polynomials. This means subtracting each term of a polynomial, which requires changing the sign of each term in a polynomial. Recall that changing the sign of 3 gives [latex]−3[/latex], and changing the sign of [latex]−3[/latex] gives 3. Just as changing the sign of a number is found by multiplying the number by [latex]−1[/latex], we can change the sign of a polynomial by multiplying it by [latex]−1[/latex]. Think of this in the same way as you would the distributive property. You are distributing [latex]−1[/latex] to each term in the polynomial. Changing the sign of a polynomial is also called finding the opposite. Be careful when there are negative terms or subtractions in the polynomial already. Just remember that you are changing the sign, so if it is negative, it will become positive. Notice that in finding the opposite of a polynomial, you change the sign of each term in the polynomial, then rewrite the polynomial with the new signs on each term. Subtract polynomials When you subtract one polynomial from another, you will first find the opposite of the polynomial being subtracted, then combine like terms. The easiest mistake to make when subtracting one polynomial from another is to forget to change the sign of EVERY term in the polynomial being subtracted. When polynomials include a lot of terms, it can be easy to lose track of the signs. Be careful to transfer them correctly, especially when subtracting a negative term. When you have many terms, like in the example above, try the vertical approach from the previous page to keep your terms organized. However you choose to combine polynomials is up to you—the key point is to identify like terms, keep track of their signs, and be able to organize them accurately. In the following video, you will see more examples of subtracting polynomials. Find the product of monomials Multiplying polynomials involves applying the rules of exponents and the distributive property to simplify the product. Polynomial multiplication can be useful in modeling real world situations. Understanding polynomial products is an important step in learning to solve algebraic equations involving polynomials. There are many, varied uses for polynomials including the generation of 3D graphics for entertainment and industry, as in the image below. Surfaces made from polynomials with AutoCAD In the exponents section,
we practiced multiplying monomials together, like we did with this expression: [latex]24{x}^{8}2{x}^{5}[/latex]. The only thing different between that section and this one is that we called it simplifying, and now we are calling it polynomial multiplication. Remember that simplifying a mathematical expression means performing as many operations as we can until there are no more to perform, including multiplication. In this section we will show examples of how to multiply more than just monomials. We will multiply monomials with binomials and trinomials. We will also learn some techniques for multiplying two binomials together. That's it! When multiplying monomials, multiply the coefficients together, and then multiply the variables together. Remember, if two variables have the same base, follow the rules of exponents, like this: [latex] \displaystyle 5{{a}^{4}}\cdot 7{{a}^{6}}=35{{a}^{10}}[/latex] The following video provides more examples of multiplying monomials with different exponents. Find the product of polynomials and monomials The distributive property can be used to multiply a monomial and a binomial. Just remember that the monomial must be multiplied by each term in the binomial. In the next example, you will see how to multiply a second degree monomial with a binomial. Note the use of exponent rules. Now let's add another layer by multiplying a monomial by a trinomial. Consider the expression [latex]2x\left(2x^{2}+5x+10\right)[/latex]. This expression can be modeled with a sketch like the one below. The only difference between this example and the previous one is there is one more term to distribute the monomial to. [latex]\begin{array}{c}2x\left(2x^{2}+5x+10\right)=2x\left(2x^{2}\right)+2x\left(5x\right)=2x\left(10\right)\\=4x^{3}+10x^{2}+20x\end{array}[/latex] You will always need to pay attention to negative signs when you are multiplying. Watch what happens to the sign on the terms in the trinomial when it is multiplied by a negative monomial in the next example. The following video provides more examples of multiplying a monomial and a polynomial. Find the product of two binomials Now let's explore multiplying two binomials. For those of you that use pictures to learn, you can draw an area model to help make sense of the process. You'll use each binomial as one of the dimensions of a rectangle, and their product as the area. The model below shows [latex]\left(x+4\right)\left(x+2\right)[/latex]: Visual representation of multiplying two binomials. Each binomial is expanded into variable terms and constants, [latex]x+4[/latex], along the top of the model and [latex]x+2[/latex] along the left side. The product of each pair of terms is a colored rectangle. The total area is the sum of all of these small rectangles, [latex]x^{2}+2x+4x+8[/latex], If you combine all the like terms, you can write the product, or area, as [latex]x^{2}+6x+8[/latex]. You can use the distributive property to determine the product of two binomials. Look back at the model above to see where each piece of [latex]x^{2}+2x+4x+8[/latex] comes from. Can you see where you multiply [latex]x[/latex] by [latex]x + 2[/latex], and where you get [latex]x^{2}[/latex] from [latex]x\left(x\right)[/latex]? Another way to look at multiplying binomials is to see that each term in one binomial is multiplied by each term in the other binomial. Look at the example above: the [latex]x[/latex] in [latex]x+4[/latex] gets multiplied by both the [latex]x[/latex] and the 2 from [latex]x+2[/latex], and the 4 gets multiplied by both the [latex]x[/latex] and the 2. The following video provides an example of multiplying two binomials using an area model as well as repeated distribution. In the next section we will explore other methods for multiplying two binomials, and become aware of the different forms that binomials can have. FOIL Foil Crane In the last section we finished with an example of multiplying two binomials,[latex]\left(x+4\right)\left(x+2\right)[/latex]. In this section we will provide examples of how to use two different methods to multiply to binomials. Keep in mind as you read through the page that simplify and multiply are used interchangeably. Some people use the FOIL method to keep track of which pairs of terms have been multiplied when you are multiplying two binomials. This is not the same thing you use to wrap up leftovers, but an acronym for First, Outer, Inner, Last. Let's go back to the example from the previous page, where we were asked to multiply the two binomials: [latex]\left(x+4\right)\left(x+2\right)[/latex]. The following steps show you how to apply this method to multiplying two binomials. [latex]\begin{array}{l}\text{First}\text{ term in each binomial}:
\,\,\,\,\,\,\,\,\,\,\,\,\,\,\left(x+4\right)\left(x+2\right)\,\,\,\,\,\,\,\,\,\,\,\,\,\,x\left(x\right)=x^{2}\\\text{Outer terms}:\,\left(x+4\right)\left(x+2\right)\,\,\,\,\,\,\,\,\,\,\,\,\,\,x\left(2\right)=2x\\\text{Inner terms}:\,\left(x+4\right)\left(x+2\right)\,\,\,\,\,\,\,\,\,\,\,\,\,\,4\left(x\right)=4x\\\text{Last terms in each binomial}:\,\,\,\,\,\,\,\,\,\,\,\,\,\,\left(x+4\right)\left(x+2\right)\,\,\,\,\,\,\,\,\,\,\,\,\,\,4\left(2\right)=8\end{array}[/latex] When you add the four results, you get the same answer, [latex]x^{2}+2x+4x+8=x^{2}+6x+8[/latex]. The last step in multiplying polynomials is to combine like terms. Remember that a polynomial is simplified only when there are no like terms remaining. Caution! Note that the FOIL method only works for multiplying two binomials together. It sill not work for multiplying a binomial and a trinomial, or two trinomials. Order Doesn't Matter When You Multiply One of the neat things about multiplication is that terms can be multiplied in either order. The expression [latex]\left(x+2\right)\left(x+4\right)[/latex] has the same product as [latex]\left(x+4\right)\left(x+2\right)[/latex], [latex]x^{2}+6x+8[/latex]. (Work it out and see.) The order in which you multiply binomials does not matter. What matters is that you multiply each term in one binomial by each term in the other binomial. Polynomials can take many forms. So far we have seen examples of binomials with variable terms on the left and constant terms on the right, such as this binomial [latex]\left(2r-3\right)[/latex]. Variables may also be on the right of the constant term, as in this binomial [latex]\left(5+r\right)[/latex]. In the next example, we will show that multiplying binomials in this form requires one extra step at the end. In the next example, you will see that sometimes there are constants in front of the variable. They will get multiplied together just as we have done before. The video that follows gives another example of multiplying two binomials using the FOIL acronym. Remember this method only works when you are multiplying two binomials. The Table Method You may see a binomial multiplied by itself written as [latex]{\left(x+3\right)}^{2}[/latex] instead of [latex]\left(x+3\right)\left(x+3\right)[/latex]. To find this product, let's use another method. We will place the terms of each binomial along the top row and first column of a table, like this: [latex]x[/latex] [latex]+3[/latex] [latex]x[/latex] [latex]+3[/latex] Now multiply the term in each column by the term in each row to get the terms of the resulting polynomial. Note how we keep the signs on the terms, even when they are positive, this will help us write the new polynomial. [latex]x[/latex] [latex]+3[/latex] [latex]x[/latex] [latex]x\cdot{x}=x^2[/latex] [latex]3\cdot{x}=+3x[/latex] [latex]+3[/latex] [latex]x\cdot{3}=+3x[/latex] [latex]3\cdot{3}=+9[/latex] Now we can write the terms of the polynomial from the entries in the table: [latex]\left(x+3\right)^{2}[/latex] = [latex]x^2[/latex] + [latex]3x[/latex] + [latex]3x[/latex] + [latex]9[/latex] = [latex]x^{2}[/latex] + [latex]6x[/latex] + [latex]9[/latex]. Pretty cool, huh? So far, we have shown two methods for multiplying two binomials together. Why are we focusing so much on binomials? They are one of the most well studied and widely used polynomials, so there is a lot of information out there about them. In the previous example, we saw the result of squaring a binomial that was a sum of two terms. In the next example we will find the product of squaring a binomial that is the difference of two terms. Caution! It is VERY important to remember the caution from the exponents section about squaring a binomial: You can't move the exponent into a grouped sum because of the order of operations!!!!! INCORRECT: [latex]\left(2+x\right)^{2}eq2^{2}+x^{2}[/latex] CORRECT: [latex]\left(2+x\right)^{2}=\left(2+x\right)\left(2+x\right)[/latex] In the video that follows, you will see another examples of using a table to multiply two binomials. Further Examples The next couple of examples show you some different forms binomials can take. In the first, we will square a binomial that has a coefficient in front of the variable, like the product in the first example on this page. In the second we will find the product of two binomials that have the variable on the right instead of the left. We will use both the FOIL method and the table method to simplify. In the last example, we want to show you another common form a binomial can take, each of the terms in the two binomials is the same, but the signs are different. You will see that in this case, the middle term will disappear. There are predictable outcomes when you square a binomial sum or difference. In general terms, for a binomial difference, [latex]\left(a-b\right)^{2}=\left(a-b\right)\left(a-b\right)[/latex], the resulting product, after being simplified, will look like this: [latex]a^2-2ab+b^2[/latex]. The product of a binomial sum will have the following predictable outcome: [latex]\left(a+b\right)^{2}=\left(a+b\right)\left(a+b\right)=a^2+2ab+b^2[/latex]. Note that a and b in these generalizations could be integers, fractions, or variables with any kind of constant. You will learn more about predictable patterns from products of binomials in later math classes. In this section we showed two ways to find the product of two binomials, the FOIL method, and by using a table. Some of the forms a product of two binomials can take are listed here: [latex]\left(x+5\right)\left(2x-3\right)[/latex] [latex]\left(x+7\right)^{2}[/latex] [latex]\left(x-1\right)^{2}[/latex] [latex]\left(2-y\right)\left(5+y\right)[/latex] [latex]\left(x+9\right)\left(x-9\right)[/latex] [latex]\left(2x-4\right)\left(x+3\right)[/latex] And this is just a small list, the possible combinations are endless. For each of the products in the list, using one of the two methods presented here will work to simplify. Divide a polynomial by a monomial The fourth arithmetic operation is division, the inverse of multiplication. Division of polynomials isn't much different from division of numbers. In the exponential section, you were asked to simplify expressions such as: [latex]\displaystyle\frac{{{a}^{2}}{{({{a}^{5}})}^{3}}}{8{{a}^{8}}}[/latex]. This expression is the division of two monomials. To simplify it, we divided the coefficients and then divided the variables. In this section we will add another layer to this idea by dividing polynomials by monomials, and by binomials. The distributive property states that you can distribute a factor that is being multiplied by a sum or difference, and likewise you can distribute a divisor that is being divided into a sum or difference. In this example, you can add all the terms in the numerator, then divide by 2. [latex]\frac{\text{dividend}\rightarrow}{\text{divisor}\rightarrow}\,\,\,\,\,\, \frac{8+4+10}{2}=\frac{22}{2}=11[/latex] Or you can first divide each term by 2, then simplify the result. [latex] \frac{8}{2}+\frac{4}{2}+\frac{10}{2}=4+2+5=11[/latex] Either way gives you the same result. The second way is helpful when you can't combine like terms in the numerator. Let's try something similar with a binomial. Divide. [latex]\frac{9a^3+6a}{3a^2}[/latex] In the next example, you will see that the same ideas apply when you are dividing a trinomial by a monomial. You can distribute the divisor to each term in the trinomial and simplify using the rules for exponents. As we have throughout the course, simplifying with exponents includes rewriting negative exponents as positive. Pay attention to the signs of the terms in the next example, we will divide by a negative monomial. Now, we ask you to think about what would happen if you were given a quotient like this to simplify: [latex] \frac{27{{y}^{4}}+6{{y}^{2}}-18}{-6y+3}[/latex]. You may be tempted to divide each term of [latex] {27{{y}^{4}}+6{{y}^{2}}-18}[/latex] individually by [latex]-6y[/latex], then [latex]3[/latex]. This would go against the order of operations because the division sign is a grouping symbol, and the addition in the denominator cannot be simplified anymore. The result is that no further operations can be performed with the tools we know. We can, however, call into use a tool that you may have learned in gradeschool: long division. Polynomial Long Division Recall how you can use long division to divide two whole numbers, say 900 divided by 37. First, you would think about how many 37s are in 90, as 9 is too small. (Note: you could also think, how many 40s are there in 90.) There are two 37s in 90, so write 2 above the last digit of 90. Two 37s is 74; write that product below the 90. Subtract: [latex]90–74[/latex] is 16. (If the result is larger than the divisor, 37, then you need to use a larger number for the quotient.) Bring down the next digit (0) and consider how many 37s are in 160. There are four 37s in 160, so write the 4 next to the two in the quotient. Four 37s is 148; write that product below the 160. Subtract: [latex]160–148[/latex] is 12. This is less than 37 so the 4 is correct. Since there are no more digits in the dividend to bring down, you're done. The final answer is 24 R12, or [latex]24\frac{12}{37}[/latex]. You can check this by multiplying the quotient (without the remainder) by the divisor, and then adding in the remainder. The result should be the dividend: [latex]24\cdot37+12=888+12=900[/latex] To divide polynomials, use the same process. This example shows how to do this when dividing by a binomial. Check this by multiplying: [latex]\left(x-6\right)\left(x+2\right)=x^{2}+2x-6x-12=x^{2}-4x-12[/latex] Polynomial long division involves many steps. Hopefully this video will help you determine what step to do next when you are using polynomial long division. Let's try another example. In this example, a term is "missing" from the dividend. Check the result: [latex]\left(x–3\right)\left(x^{2}+3x+3\right)\,\,\,=\,\,\,x\left(x^{2}+3x+3\right)–3\left(x^{2}+3x+3\right)\\\,=\,\,\,x^{3}+3x^{2}+3x–3x^{2}–9x–9\\\,=\,\,\,x^{3}–6x–9\\\,\,\,\,\,\,\,\,\,\,x^{3}–6x–9+\left(-1\right)\,\,\,=\,\,\,x^{3}–6x–10[/latex] The video that follows shows another example of dividing a third degree trinomial by a first degree binomial. The last video example shows how to divide a third degree trinomial by a second degree binomial. Summary To divide a monomial by a monomial, divide the coefficients (or simplify them as you would a fraction) and divide the variables with like bases by subtracting their exponents. To divide a polynomial by a monomial, divide each term of the polynomial by the monomial. Be sure to watch the signs! Final answers should be written without any negative exponents. Dividing polynomials by polynomials of more than one term can be done using a process very much like long division of whole numbers. You must be careful to subtract entire expressions, not just the first term. Stop when the degree of the remainder is less than the degree of the divisor. The remainder can be written using R notation, or as a fraction added to the quotient with the remainder in the numerator and the divisor in the denominator. Summary We have seen that subtracting a polynomial means changing the sign of each term in the polynomial and then reorganizing all the terms to make it easier to combine those that are alike. How you organize this process is up to you, but we have shown two ways here. One method is to place the terms next to each other horizontally, putting like terms next to each other to make combining them easier. The other method was to place the polynomial being subtracted underneath the other after changing the signs of each term. In this method it is important to align like terms and use a blank space when there is no like term. Multiplication of binomials and polynomials requires an understanding of the distributive property, rules for exponents, and a keen eye for collecting like terms. Whether the polynomials are monomials, binomials, or trinomials, carefully multiply each term in one polynomial by each term in the other polynomial. Be careful to watch the addition and subtraction signs and negative coefficients. A product is written in simplified form if all of its like terms have been combined.